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THE PERIODIC MOTIONS OF A NON-AUTONOMOUS HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM AT PARAMETRIC RESONANCE OF THE FUNDAMENTAL TYPE†

O. V. KHOLOSTOVA

Moscow

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The motion of an almost autonomous Hamiltonian system with two degrees of freedom, 2π-periodic in time, is considered. It is assumed that the origin is an equilibrium position of the system, the linearized unperturbed system is stable, and its characteristic exponents $\pm i\omega j$ ($j = 1, 2$) are pure imaginary. In addition, it is assumed that the number $2\omega_1$ is approximately an integer, that is, the system exhibits parametric resonance of the fundamental type. Using Poincaré's theory of periodic motion and KAM-theory, it is shown that 4π-periodic motions of the system exist in a fairly small neighbourhood of the origin, and their bifurcation and stability are investigated. As applications, periodic motions are constructed in cases of parametric resonance of the fundamental type in the following problems: the plane elliptical restricted three-body problem near triangular libration points, and the problem of the motion of a dynamically symmetrical artificial satellite near its cylindrical precession in an elliptical orbit of small eccentricity.

1. FORMULATION OF THE PROBLEM. TRANSFORMATION OF THE HAMILTONIAN

Consider the motion of an almost autonomous Hamiltonian system with two degrees of freedom. It will be assumed that the Hamiltonian of the system is expressed as a series in powers of a small parameter ε ($0 < \varepsilon \ll 1$)

$$H = H_0(q_i, p_i) + \varepsilon H_1(q_i, p_i, t) + \varepsilon^2 H_2(q_i, p_i, t) + \dots \tag{1.1}$$

where q_i and p_i ($i = 1, 2$) are the coordinates and momenta, respectively. The functions $H_k(q_i, p_i, t)$ ($k = 1, 2, \dots$) in (1.1) are assumed to be 2π-periodic functions of time.

Suppose the origin $q_i = p_i = 0$ of the phase space is an equilibrium position of the system. The Hamiltonian H is assumed to be analytic in the neighbourhood of the point $q_i = p_i = 0$; the functions H_k ($k = 1, 2, \dots$) can be represented in the form

$$H_k = H_k^{(2)} + H_k^{(3)} + H_k^{(4)} + \dots \tag{1.2}$$

where $H_k^{(l)}$ is a form of degree l in q_i, p_i.

Let us assume that the characteristic exponents $\pm i\omega j (j = 1, 2)$ of the system of equations of motion, linearized in the neighbourhood of the origin, are pure imaginary where $\varepsilon = 0$. If the numbers ω_j do not satisfy any relations of the form $k_1\omega_1 + k_2\omega_2 = 0$ (where k_1 and k_2 are integers such that $1 \leqslant |k_1| + |k_2| \leqslant 4$), then, for suitably chosen variables q_i, p_i, the unperturbed Hamiltonian H_0 may be written in normal form up to and including fourth-order terms. In "polar" coordinates φ_i, r_i ($q_i = \sqrt{2r_i} \sin \varphi_i, p_i = \sqrt{2r_i} \cos \varphi_i$), we have

$$H_0 = \lambda_1 r_1 + \lambda_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + O_5$$

where $\lambda_1 = \omega_1$, but $\lambda_2 = \omega_2$ or $\lambda_2 = -\omega_2$ (depending on the specific problem concerned), c_{ij} are constants, and O_5 is the set of terms of order at least five in $r_i^{1/2}$.

Suppose parametric resonance of the fundamental type occurs in the system when $2\omega_1$ is approximately an odd integer N.

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The aim of this investigation is to determine whether periodic motions of the complete system with Hamiltonian (1.1), (1.2) exist in a fairly small neighbourhood of the origin, and to determine the number and stability of such motions.

We shall assume that besides the resonance relation $2\omega_1 \approx N$ there are no other relations between the frequencies ω_1 and ω_2 of the form $k_1\omega_1 + k_2\omega_2 \approx L$ (where k_i and L are integers, with $2 \leqslant |k_1|+|k_2| \leqslant 4$). We set $r_i = \varepsilon R_i$, $\varphi_i = \theta_i$ $(i = 1, 2)$ and, applying a nearly identical canonical transformation which is 2π-periodic with respect to time, we reduce the Hamiltonian to the form

$$H = \tilde{\lambda}_1 R_1 + \tilde{\lambda}_2 R_2 + \varepsilon\sigma R_1 \cos(2\theta_1 - Nt + \theta_0) + \varepsilon(c_{20}R_1^2 + c_{11}R_1R_2 + c_{02}R_2^2) + O(\varepsilon^{3/2}) \tag{1.3}$$

where $\tilde{\lambda}_i = \lambda_i + O(\varepsilon) = \text{const}$ $(i = 1, 2)$, and σ and θ_0 are constants. The constant σ in (1.3) is assumed to be positive; this may always be achieved by a displacement with respect to θ_1.

Put $2\tilde{\lambda}_1 = N + 2\varepsilon\beta$. Make the change of variables $\theta_i, R_i \to \psi_i, \rho_i$ defined by

$$R_i = \rho_i \quad (i = 1,2), \quad 2\theta_1 - Nt + \theta_0 = 2\psi_1, \quad \theta_2 = \psi_2$$

thereby reducing the Hamiltonian of the problem to the form

$$H = \varepsilon\beta\rho_1 + \tilde{\lambda}_2\rho_2 + \varepsilon\sigma\rho_1\cos 2\psi_1 + \varepsilon(c_{20}\rho_1^2 + c_{11}\rho_1\rho_2 + c_{02}\rho_2^2) + O(\varepsilon^{3/2}) \tag{1.4}$$

Assuming that $c_{20} \neq 0$, we make one more change of variables, through the formulae

$$\psi_2 = \tilde{\psi}_2, \quad \rho_1 = \frac{\sigma}{|c_{20}|}\rho, \quad \rho_2 = \frac{\sigma}{|c_{20}|}\tilde{\rho}_2 \quad (\varkappa = \operatorname{sign} c_{20})$$

We then have

$$H = \tilde{\lambda}_2\tilde{\rho}_2 + \varepsilon\{\alpha_1\tilde{\rho}_2^2 + \alpha_2[(a\beta + b\tilde{\rho}_2)\rho + \rho\cos 2\psi + \rho^2]\} + O(\varepsilon^{3/2}) \tag{1.5}$$

$$\alpha_1 = \frac{c_{02}\sigma}{|c_{20}|}, \quad \alpha_2 = \varkappa\sigma, \quad a = \alpha_2^{-1}, \quad b = \frac{c_{11}}{c_{20}}$$

The term $O(\varepsilon^{3/2})$ in (1.5) is 4π-periodic in t, 2π-periodic in ψ and $\tilde{\psi}_2$, and analytic in $\rho^{1/2}$ and $\tilde{\rho}_2^{1/2}$.

Remark. The Hamiltonian (1.1), (1.2) can also be reduced to the form (1.5) when $\omega_1 \approx N/2$ and for even N, provided that its structure contains no third-degree forms $H_k^{(3)}$. In that case the term $O(\varepsilon^{3/2})$ in (1.5) will be 2π-periodic in t.

2. PERIODIC MOTIONS OF THE SYSTEM

Equilibrium positions of the approximate system. If the term $O(\varepsilon^{3/2})$ is omitted from Hamiltonian (1.5), we obtain an approximate Hamiltonian. The coordinate $\tilde{\psi}_2$ in the corresponding system is cyclic, and consequently $\tilde{\rho}_2 = c = \text{const}$. We write the approximate Hamiltonian in the form

$$\tilde{H} = \tilde{\lambda}_2 c + \varepsilon(\alpha_1 c^2 + \alpha_2 H') \tag{2.1}$$

$$H' = -\chi\rho + \rho\cos 2\psi + \rho^2, \quad \chi = -(a\beta + bc) = \text{const} \tag{2.2}$$

The function H' is a model Hamiltonian for systems with one degree of freedom at parametric resonance (see, e.g., [1]). However, while for the latter systems the parameter χ is defined by the resonance detuning, in the case of the system considered here, which has two degrees of freedom, χ will depend not only on the resonance detuning (characterized by the parameter β) but also on the constant c, which is associated with the presence of a second (cyclic) coordinate in the system.

The model system has a particular solution $\rho = 0$ – the equilibrium position at the origin, it is stable if $|\chi| > 1$ and unstable if $|\chi| < 1$. If $\chi < -1$, there are no other equilibrium positions. If $-1 < \chi < 1$, the model system has two stable equilibrium positions – the points $(\pi/2, (\chi + 1)/2)$ and $(3\pi/2, (\chi + 1)/2)$. These points are also stable equilibrium positions for $\chi > 1$; in the latter case the system also has two unstable equilibrium positions – the points $(\pi, (\chi - 1)/2)$ and $(0, (\chi - 1)/2)$.

Phase portraits of the model system are shown in Figs 1a, b, c in the plane of the variables $x_1 = \sqrt{2\rho}\cos\psi$, $x_2 = \sqrt{2\rho}\sin\psi$, for the cases $\chi < -1$, $-1 < \chi < 1$, $\chi > 1$, respectively. Corresponding to the stable equilibrium positions of the model system in Fig. 1 there are singular points of the "centre" type; the unstable equilibria are represented by saddle points.

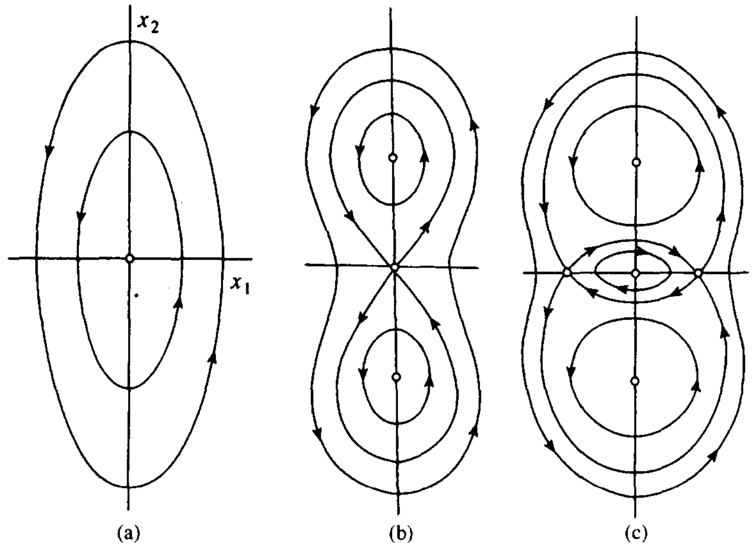

Fig. 1

The equilibrium positions of the approximate system with two degrees of freedom are given by the equalities $\tilde{\rho}_2 = 0$, $\rho = 0$, or

$$\tilde{\rho}_2 = 0, \quad \rho = \rho_*, \quad \psi = \psi_* \tag{2.3}$$

where (ψ_*, ρ_*) is one of the equilibrium positions not coinciding with the origin. For these equilibrium points $c = 0$, and so the parameter χ of the model system and hence also the equilibrium values $\rho_* = (\chi \pm 1)/2$, are defined only by the resonance detuning.

Periodic motions of the complete system. Let us consider the equilibrium position (2.3) of the approximate system as a generating solution. Setting $\tilde{\rho}_2 = r_2$, $\psi = \psi_* + x_1$, $\rho = \rho_* + y$ in (1.5), we obtain the following expression for the complete Hamiltonian in the neighbourhood of the generating solution

$$H = \Lambda_2 r_2 + \varepsilon\alpha_2(-2\rho_* \cos 2\psi_* x_1^2 + y_1^2) + \varepsilon \cdot O_3 + O(\varepsilon^{3/2}) \tag{2.4}$$

$$\Lambda_2 = \tilde{\lambda}_2 + \varepsilon\alpha_2 b\rho_*$$

where O_3 is the set of terms of degree at least three in $x_1, y_1, r_2^{1/2}$ with constant coefficients and the term $O(\varepsilon^{3/2})$ is 4π-periodic in t.

Since by assumption $2\omega_2$ is not close to an integer, we have non-resonant case of Poincaré's theory of periodic motions [2], so that each equilibrium position (2.3) of the approximate system generates a single solution of the complete system, 4π-periodic in t and analytic in $\varepsilon^{1/2}$.

$$\rho = \tilde{\rho}(t) = \rho_* + O(\varepsilon^{1/2}), \quad \psi = \tilde{\psi}(t) = \psi_* + O(\varepsilon^{1/2}), \quad \tilde{\rho}_2 = \tilde{\rho}_2(t) = O(\varepsilon) \tag{2.5}$$

Depending on the value of the parameter χ, the number of such periodic solutions may be either four (if $\chi > 1$), two (if $-1 < \chi < 1$) or zero (if $\chi < -1$).

Corresponding to the solutions (2.5) we have the following motions of the original system with Hamiltonian (1.1), which are 4π-periodic in t.

$$q_1(t) = \sqrt{\frac{2\varepsilon\sigma\rho_*}{|c_{20}|}} \sin\left(\frac{Nt}{2} + \psi_* + \frac{\pi}{2}(1-\varkappa) - \frac{\theta_0}{2}\right) + O(\varepsilon) \tag{2.6}$$

$$p_1(t) = \sqrt{\frac{2\varepsilon\sigma\rho_*}{|c_{20}|}} \cos\left(\frac{Nt}{2} + \psi_* + \frac{\pi}{2}(1-\varkappa) - \frac{\theta_0}{2}\right) + O(\varepsilon)$$

$$q_2 = O(\varepsilon), \quad p_2 = O(\varepsilon)$$

Motions (2.6), corresponding to equilibrium positions of the model system whose ψ_* values differ from one another by π, are obtained from one another by a time shift of $2\pi/N$. Hence the original system has two different periodic motions of type (2.6) for $\chi > 1$, one motion for $-1 < \chi < 1$, and none for $\chi < -1$.

The stability of the periodic motions. We will now consider the stability of the periodic solutions determined above.

Motions corresponding to unstable equilibrium positions of the model system are unstable, as follows from the fact that the characteristic equation of the linearized approximate system has a positive real root.

To solve the problem of the stability of periodic motions corresponding to stable equilibrium positions of the model system, we will consider the Hamiltonian of the perturbed motion, making the following substitutions in (1.5)

$$\tilde{\rho}_2 = r_2, \quad \psi = \tilde{\psi}(t) + x_1, \quad \rho = \tilde{\rho}(t) + y_1$$

We have

$$H = \Lambda_2 r_2 + \varepsilon\alpha_2[(\chi+1)x_1^2 + y_1^2] + 2\varepsilon\alpha_2 x_1^2 y_1 + \varepsilon\alpha_2 b y_1 r_2 -$$

$$-\tfrac{1}{3}\varepsilon\alpha_2(\chi+1)x_1^4 + \varepsilon\alpha_1 r_2^2 + \varepsilon \cdot O_5 + O(\varepsilon^{3/2}) \tag{2.7}$$

where O_5 is the set of terms of at least the fifth degree in $x_1, y_1, r_2^{1/2}$, the term $O(\varepsilon^{3/2})$ is 4π-periodic in t, and the constant Λ_2 is defined by (2.4) with $\rho_* = (\chi + 1)/2$.

Making the change of variables

$$x_1 = x(\chi+1)^{-1/4}, \quad y_1 = y(\chi+1)^{1/4}, \quad r_2 = r_2$$

we then apply a nearly identical canonical transformation

$$x, y, \tilde{\psi}_2, r_2 \to x^*, y^*, \psi_2^*, r_2^*$$

which normalizes the Hamiltonian up to terms of fourth order inclusive. This transformation may be obtained, for example, by using the Deprit-Hori method [3]. We then change from the variables x^* and y^* to "polar" coordinates ψ^* and r^*, in accordance with the formulae

$$x^* = \sqrt{2r^*}\sin\psi^*, \quad y^* = \sqrt{2r^*}\cos\psi^*$$

as a result of which the transformed Hamiltonian becomes

$$H = \Lambda_2 r_2^* + 2\varepsilon\alpha_2\sqrt{\chi+1}\,r^* + \varepsilon(C_{20}r^{*2} + C_{11}r^* r_2^* + C_{02}r_2^{*2}) + \varepsilon\cdot O_5 + O(\varepsilon^{3/2}) \tag{2.8}$$

$$C_{20} = -\frac{\alpha_2(\chi+4)}{2(\chi+1)}, \quad C_{11} = -\frac{c_{11}\sigma}{|c_{20}|\sqrt{\chi+1}}, \quad C_{02} = \frac{\alpha_2(4c_{02}c_{20} - c_{11}^2)}{4c_{20}^2}$$

If the condition $C_{11}^2 - 4C_{02}C_{20} \neq 0$ is satisfied, the periodic solution in question is stable for the majority of initial data [3, 4]. This last relation reduces to an inequality

$$2c_{11}^2 + (\chi+4)(4c_{20}c_{02} - c_{11}^2) \neq 0 \tag{2.9}$$

Thus, if $-1 < \chi < 1$, the unique 4π-periodic motion of the system with Hamiltonian (1.1) is stable for the majority of initial data, provided that condition (2.9) holds; of the two periodic motions for $\chi > 1$, one is unstable and one stable for the majority of initial data (provided condition (2.9) holds).

3. PERIODIC MOTIONS NEAR TRIANGULAR LIBRATION POINTS OF THE PLANE ELLIPTICAL RESTRICTED THREE-BODY PROBLEM

We shall construct periodic motions in the neighbourhood of triangular libration points of the plane, elliptical restricted three-body problem. The Hamiltonian is [3]

$$H=\frac{1}{2}(p_\xi^2+p_\eta^2)+p_\xi\eta-p_\eta\xi+\frac{e\cos\nu}{2(1+e\cos\nu)}(\xi^2+\eta^2)-\frac{W}{1+e\cos\nu}$$

$$W=\frac{1-\mu}{r_1}+\frac{\mu}{r_2},\quad \mu=\frac{m_2}{m_1+m_2},\quad r_1=\sqrt{(\xi+\mu)^2+\eta^2},\quad r_2=\sqrt{(\xi+\mu-1)^2+\eta^2} \tag{3.1}$$

where ξ, η and p_ξ, p_η are the Nechvil variables and the corresponding generalized momenta, e is the eccentricity, ν is the true anomaly, m_1 and m_2 are the masses of the main attracting bodies.

The system with Hamiltonian (3.1) has a particular solution

$$\varepsilon_0=\frac{1-2\mu}{2},\quad \eta_0=\frac{\sqrt{3}}{2},\quad p_{\xi_0}=-\frac{\sqrt{3}}{2},\quad p_{\eta_0}=\frac{1-2\mu}{2} \tag{3.2}$$

corresponding to a triangular libration point. At $e = 0$ a necessary condition for solution (3.2) to be stable is the inequality

$$0<\mu<\mu_*=(9-\sqrt{69})/18=0.0385208\ldots$$

Let us consider the motions of the system in the neighbourhood of the point (3.2). In (3.1) we put

$$\xi=\xi_0+q_1,\quad \eta=\eta_0+q_2,\quad p_\xi=p_{\xi_0}+p_1,\quad p_\eta=p_{\eta_0}+p_2$$

We then obtain [3]

$$H=H_2+H_3+H_4+\ldots \tag{3.3}$$

$$H_2=\frac{1}{2}(p_1^2+p_2^2)+p_1q_2-q_1p_2+\frac{e\cos\nu}{2(1+e\cos\nu)}(q_1^2+q_2^2)+$$

$$+\frac{1}{8(1+e\cos\nu)}(q_1^2-8kq_1q_2-5q_2^2),\quad k=\frac{3\sqrt{3}(1-2\mu)}{4}$$

where H_3 and H_4 are forms of third and fourth degree in q_i and p_i ($i = 1, 2$), which will not be shown here.

Using a univalent linear canonical transformation $q_i, p_i \to q_i', p_i'$ [3], we reduce the quadratic part H_2 of Hamiltonian (3.3) at $e = 0$ to normal form. The frequencies ω_1 and ω_2 ($\omega_1 > \omega_2 > 0$) of small oscillations satisfy the equation

$$\omega^4-\omega^2+\frac{27}{4}\mu(1-\mu)=0$$

When $\mu = \mu_0 = (3-2\sqrt{2})/6 = 0.0285954\ldots$, we have $\omega_2 = 1/2$, that is, parametric resonance of the main type occurs in the system. Assuming that $0 < e \ll 1$, we find the 4π-periodic motions in the case when $\omega_2 \approx 1/2$.

We put $q_i' = \tilde{q}_i/\sqrt{\omega_i}$, $p_i' = \sqrt{\omega_i}\tilde{p}_i$ and then normalize the complete form H_2 in the terms of the order of e, as well as the forms H_3 and H_4 at $e = 0$. After changing to "polar" coordinates we obtain the Hamiltonian

$$H=\tilde{\omega}_1R_1-\tilde{\omega}_2R_2+e\sigma R_2\cos(2\theta_2+\nu+\theta_0)+e(c_{20}R_1^2+c_{11}R_1R_2+c_{02}R_2^2)+O(e^{3/2})$$

which, apart from the notation, is identical with Hamiltonian (1.3).

Here, as calculations will show,

$$\tilde{\omega}_1=\frac{\sqrt{3}}{2}+O(e),\quad \tilde{\omega}_2=\frac{1}{2}+9\sqrt{2}(\mu-\mu_0)+O(e^2),\quad \sigma=\frac{\sqrt{33}}{8},\quad \theta_0=\operatorname{arctg}\frac{4\sqrt{6}}{27}$$

$$c_{20}=\frac{15}{16},\quad c_{11}=\frac{40\sqrt{3}}{3},\quad c_{02}=\frac{341}{48}$$

When determining the numerical values of the coefficients c_{ij}, we used their expressions as functions of the frequencies [5].

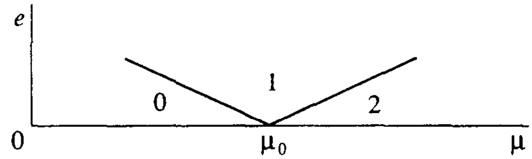

Fig. 2

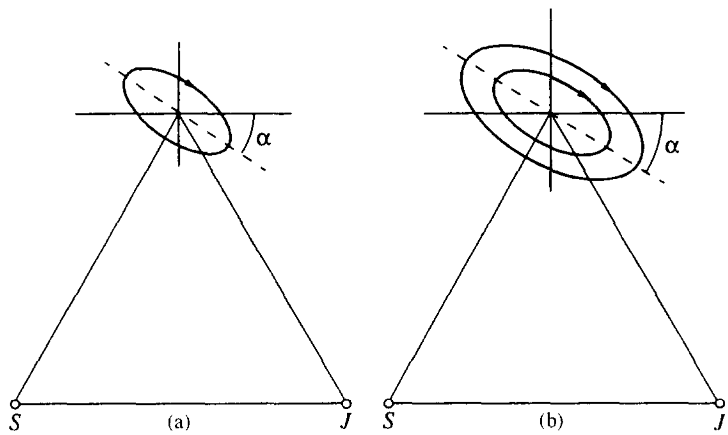

Fig. 3

Introducing the frequency detuning $\tilde{\omega}_1 = \frac{1}{2} + e\sigma\chi$ and relying on the results of Sections 1 and 2, we obtain the following 4π-periodic motions in the neighbourhood of a triangular libration point

$$\xi = \xi_0 + 2\sqrt{e}a_* \sin\tau + O(e), \quad \eta = \eta_0 - 0.4\sqrt{e}a_*(\sqrt{6}\sin\tau + 2\cos\tau) + O(e) \tag{3.4}$$

$$a_* = \sqrt{\frac{30\sqrt{33}}{341}\rho_*}, \quad \tau = \psi_* - \frac{\nu}{2} - \frac{\theta_0}{2}$$

where (ψ_*, ρ_*) is an equilibrium position of the model system (see Section 1.2).

Relations (3.4) (omitting terms $O(e)$) are the equations of an ellipse. The major axis of the ellipse is inclined to the $\eta = 0$ axis at an angle $\alpha = -0.5 \operatorname{arctg}(2\sqrt{6}/3) = -29.26°\ldots$, and the quotient of the lengths of the axes of the ellipse is $(41 + 7\sqrt{33})/8 = 3.186\ldots$.

Fixing the parameters μ and e $(\mu - \mu_0 \sim e)$, we derive from the relation $\mu - \mu_0 = \sqrt{66}e\chi/144 + O(e^2)$ the corresponding value of the parameter χ of the model system, and hence the number and form of the periodic solutions (3.4).

Regions 0, 1 and 2 in the plane of the parameters (μ, e) in the neighbourhood of the point $\mu = \mu_0$, $e = 0$, as shown in Fig. 2, correspond to the cases $\chi < -1$, $-1 < \chi < 1$ and $\chi > 1$. The boundary curves between the regions are given by the equations $\mu = \mu_0 \pm \sqrt{66}e/144 + O(e^2)$; on these curves the parameter χ takes values ± 1. In region 0 there are no 4π-periodic motions of the system in the neighbourhood of a triangular libration point. In region 1 one 4π-periodic motion of the form (3.4) exists, which is stable for the majority of initial data. In region 2, two motions (3.4) exist, one of which is stable (for the majority of initial data) and one unstable. Condition (2.9) for stable motions is always satisfied in the region $\chi > -1$ in which these motions exist.

Periodic motions of regions 1 and 2 are shown in Fig. 3(a, b). Motions in elliptic orbits occur in the sense opposite to that of the rotation of the body J about the body S. In the case when two periodic motions exist (Fig. 3b), the outer ellipse corresponds to the stable motion and the inner one to the unstable motion.

4. PERIODIC MOTIONS OF A DYNAMICALLY SYMMETRIC ARTIFICIAL SATELLITE, CLOSE TO CYLINDRICAL PRECESSION

We will now consider the motion of a dynamically symmetric artificial satellite – a rigid body in a central Newtonian gravitational field in an elliptic orbit of small eccentricity. Assuming that the dimensions of the satellite are small compared with those of the orbit, we assume, as usual, that the motion of the satellite about its centre of mass is independent of the motion of the centre of mass itself.

We introduce an orbital system of coordinates *OXYZ* (the *OX* axis points along the transversal to the orbit, the *OY* axis points along the binormal, and the *OZ* axis points along the radius vector of the centre of mass *O* of the satellite) and a system of coordinates *Oxyz* attached to the satellite, with the *Oz* axis pointing along the satellite's axis of symmetry. The orientation of the attached system of coordinates relative to the orbital system is defined by the Euler angles ψ, θ and φ.

A motion of the satellite exists in which its axis of symmetry is perpendicular to the orbital plane throughout the motion, while the satellite itself rotates about the axis of symmetry at constant angular velocity (cylindrical precession). When that is the case, $\theta_0 = \pi/2$, $\psi_0 = \pi$, and the momenta canonically conjugate to θ and ψ, p_0 and p_ψ, take zero values.

Using the Hamiltonian as presented in [6] and putting

$$\theta = \pi/2 + q_1, \quad \psi = \pi + q_2, \quad p_\theta = p_1, \quad p_\psi = p_2$$

we obtain the Hamiltonian of the perturbed motion of the satellite near its cylindrical precession:

$$H = H_2 + H_4 + \dots \tag{4.1}$$

$$H_2 = \frac{p_1^2 + p_2^2}{2(1+e\cos\nu)^2} - p_2 q_1 + \frac{\alpha\beta(1-e^2)^{3/2}}{(1+e\cos\nu)^2} q_1 p_2 + p_1 q_2 - \frac{1}{2}\alpha\beta(1-e^2)^{3/2}(q_1^2 - q_2^2) +$$

$$+ \frac{\alpha^2\beta^2(1-e^2)^3}{2(1+e\cos\nu)^2} q_1^2 + \frac{3}{2}(\alpha-1)(1+e\cos\nu)q_1^2$$

$$H_4 = \left[\frac{1}{2}\alpha^2\beta^2 - \frac{5}{24}\alpha\beta + \frac{1}{2}(1-\alpha)\right]q_1^4 + \left(\frac{5}{6}\alpha\beta - \frac{1}{3}\right)p_2 q_1^3 - \frac{\alpha\beta}{24}q_2^4 - \frac{1}{6}p_1 q_2^3 +$$

$$+ \frac{1}{2}p_2^2 q_1^2 + \frac{\alpha\beta}{4}q_1^2 q_2^2 + \frac{1}{2}p_2 q_1 q_2^2 + O(e); \quad \alpha = \frac{C}{A} \quad (0 \leqslant \alpha \leqslant 2)$$

where e is the eccentricity of the orbit of the satellite's centre of mass, ν is the true anomaly, A and C are the equatorial and axial moments of inertia, $\beta = r_0/\omega_0$, r_0 being the projection of the absolute angular velocity of the satellite onto the axis of symmetry (r_0 = const) and ω_0 corresponds to the mean motion of the centre of mass.

The frequencies ω_1 and ω_2 ($\omega_1 > \omega_2 > 0$) of the oscillations of the system with Hamiltonian H_2 at $e = 0$ satisfy the equation

$$\omega^4 - (\alpha^2\beta^2 - 2\alpha\beta + 3\alpha - 1)\omega^2 + (\alpha\beta - 1)(\alpha\beta + 3\alpha - 4) = 0$$

The plane of the parameters (α, β) contains a denumerable set of curves on which parametric resonance of the main type occurs. We shall confine ourselves to considering three resonant cases.

Let $\beta = 0$ (corresponding to translational motion of the satellite in absolute space). Then at $\alpha = \alpha_1 = 181/156 = 1.1603\ldots$ we have $\omega_1 = 3/2$, and at $\alpha = \alpha_2 = 23/20 = 1.15$ we have $\omega_2 = 1/2$. If the parameters α and β satisfy the relation $\alpha\beta = 2$, then $\omega_1 = 1$ when $2/3 < \alpha < 1$ and $\omega_2 = 1$ when $1 < \alpha < 2$.

Following the algorithm described in Sections 1 and 2, we shall find the periodic motions of the satellite near cylindrical precession, in near-resonant cases.

First, making the linear change of variables $q_i, p_i \to q_i', p_i'$ ($i = 1, 2$), we reduce the function H_2 at $e = 0$ to normal form. The form of the change when $\beta = 0$ was indicated before in [7]. When $\alpha\beta = 2$, the change of variables is

$$q_1 = q_2'/\sqrt{\omega_2}, \quad q_2 = q_1', \quad p_1 = p_2'\sqrt{\omega_2} - q_1', \quad p_2 = p_1' - q_2'/\sqrt{\omega_2} \quad (\omega_2 = \sqrt{3\alpha - 2}) \tag{4.2}$$

if $2/3 < \alpha < 1$; but if $1 < \alpha < 2$, the variables $q_1', q_2', p_1', p_2', \omega_2$ in formulae (4.2) must be replaced by $q_2', q_1', p_2', p_1', \omega_1$, respectively.

The resonance terms in the form H_2 when $e \neq 0$, in the cases $\omega_1 \simeq 3/2$, $\omega_2 \simeq 1/2$ and $\omega_i \simeq 1$ ($i = 1, 2$), are of orders e^3, e and e^2, respectively, so that normalization of the form H_2 must be carried out up to terms of order e^3, e and e^2 inclusive.

Normalizing H_4 and changing to "polar" coordinates θ_i and R_i by the formulae

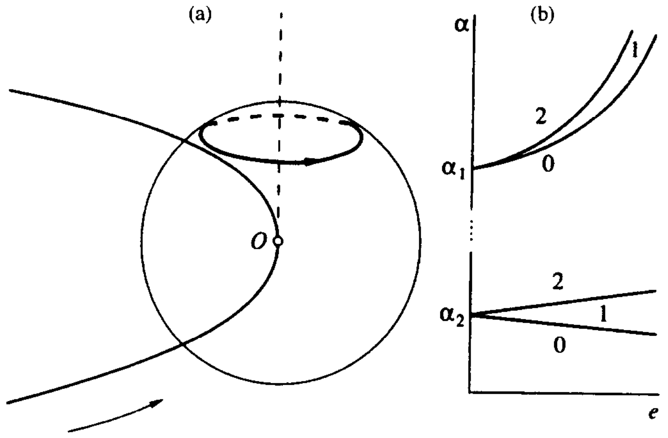

Fig. 4

$$q_i' = \sqrt{2e^k R_i}\sin\theta_i, \quad p_i' = \sqrt{2e^k R_i}\cos\theta_i$$

where $k = 3$, 1 or 2, we obtain a Hamiltonian similar to Hamiltonian (1.3) of Section 1.

When $\beta = 0$, $\omega_1 \simeq {}^3/_2$, this Hamiltonian becomes

$$H = \tilde{\omega}_1 R_1 - \tilde{\omega}_2 R_2 + e^3\sigma R_1\cos(2\theta_1 - 3\nu + \pi) + e^3(c_{20}R_1^2 + c_{11}R_1R_2 + c_{02}R_2^2) + O(e^{9/2}) \tag{4.3}$$

$$\tilde{\omega}_1 = \frac{3}{2} + \frac{169}{105}(\alpha - \alpha_1^*) + O(e^4), \quad \alpha_1^* = \alpha_1 + e^2\alpha_{12}, \quad \alpha_{12} = \frac{59751675}{13541632} = 4.4124...$$

$$\tilde{\omega}_2 = \frac{\sqrt{39}}{13} + O(e^2), \quad \sigma = \frac{23475}{2048}, \quad c_{20} = c_{02} = -\frac{25}{1764}, \quad c_{11} = -\frac{244\sqrt{39}}{1323}$$

The coefficients c_{ij} in (4.3) (and below in (4.5)) are calculated using formulae given in [7].

Introducing the resonance detuning by the formula $\tilde{\omega}_1 = {}^3/_2 + e^3\chi\sigma$ (where χ is the parameter of the model system), we obtain the following 4π-periodic motions of the satellite

$$\theta = \frac{\pi}{2} - \frac{13}{80}a_* e^{3/2}\sin\left(\frac{3\nu}{2} + \psi_*\right) + O(e^3), \quad \psi = \pi - \frac{3}{20}a_* e^{3/2}\cos\left(\frac{3\nu}{2} + \psi_*\right) + O(e^3) \tag{4.4}$$

where $a_* = \sqrt{65730\rho_*}$, and (ψ_*, ρ_*) is an equilibrium position of the model system.

Formulae (4.4) (ignoring the terms $O(e^3)$) defines a motion of the satellite in which the end of the unit vector of its axis describes a curve on the unit sphere whose projection onto the plane OXZ of the orbital system of coordinates is an ellipse with semi axes $\sim e^{3/2}$ (the ratio of the lengths of the axes is 13.12) (Fig. 4a). The satellite axis moves in the same direction as its centre of mass in motion in the orbit.

If $\beta = 0$, $\omega_2 \simeq {}^1/_2$, the normalized Hamiltonian has the form (4.3) in which e^3 is replaced by e, the resonance term by $e\sigma R_2\cos(2\theta_2 + \nu)$, and we put

$$\tilde{\omega}_1 = \frac{\sqrt{55}}{5} + O(e), \quad \tilde{\omega}_2 = \frac{1}{2} - \frac{25}{13}\left(\alpha - \frac{23}{20}\right) + O(e^2) \tag{4.5}$$

$$\sigma = \frac{3}{104}, \quad c_{20} = c_{02} = -\frac{9}{676}, \quad c_{11} = -\frac{284\sqrt{55}}{1859}$$

Introducing the resonance detuning by the formula $\tilde{\omega}_2 = {}^1/_2 - e\chi\sigma$, we obtain the following 4π-periodic motions of the satellite:

$$\theta = \frac{\pi}{2} + 5a_*\sqrt{e}\sin\left(\frac{\nu}{2} + \psi_*\right) + O(e), \quad \psi = \pi - 4a_*\sqrt{e}\cos\left(\frac{\nu}{2} + \psi_*\right) + O(e) \tag{4.6}$$

where $a_* = \sqrt{2\rho_*/3}$. The motion of the satellite corresponding to formulae (4.6) is similar to the previous motion, except that the semiaxes of the ellipse are of the order of $\sqrt{e}$, the ratio of their lengths is $^5/_4$, and the satellite axis moves in the direction opposite to that of its centre of mass in the orbit.

Regions 0, 1 and 2 in the plane of the parameters (e, α) in the neighbourhood of the points $e = 0$, $\alpha = \alpha_1$ and $e = 0$, $\alpha = \alpha_2$, as shown in Fig. 4(b), contain respectively 0, 1 and 2 periodic motions of the satellite, of the form (4.4) and (4.6). The boundaries between the regions are curves

$$\alpha = \alpha_1^* \pm \frac{2464875}{346112} e^3 + O(e^4) \quad \text{and} \quad \alpha = \alpha_2 \pm \frac{3}{200} e + O(e^2)$$

on which the parameter χ of the model system takes values ± 1.

When $\alpha\beta = 2$, $\omega_i \simeq 1$ ($i = 1$ or 2), the normalized Hamiltonian is

$$H = \tilde{\omega}_1 R_1 + \tilde{\omega}_2 R_2 + e^2 \sigma_i R_i \cos(2\theta_i - 2\nu + \theta_{0i}) + e^2 (c_{20} R_1^2 + c_{11} R_1 R_2 + c_{02} R_2^2) + O(e^3) \tag{4.7}$$

where $\theta_{01} = \pi$, $\theta_{02} = 0$, and the quantities σ_i and c_{kl} are evaluated for an arbitrary point $(\alpha_0, 2/\alpha_0)$ of the curve $\alpha\beta = 2$ ($\alpha_0 \neq 1$), with

$$\sigma_i = \tfrac{3}{2} |\alpha_0 - 1| / (3\alpha_0 - 2)$$

$$c_{20} = \tfrac{1}{8}, \quad c_{11} = 1/(2\omega_2), \quad c_{02} = (3 - 2\omega_2^2)/(8\omega_2^2), \quad \omega_2 = \sqrt{3\alpha_0 - 2} \quad \text{for } \tfrac{2}{3} < \alpha_0 < 1$$

$$c_{20} = (3 - 2\omega_1^2)/(8\omega_1^2), \quad c_{11} = 1/(2\omega_1), \quad c_{02} = \tfrac{1}{8}, \quad \omega_1 = \sqrt{3\alpha_0 - 2} \quad \text{for } 1 < \alpha_0 < 2$$

The quantity $\tilde{\omega}_i$ in (4.7) corresponding to a resonance frequency ω_i is evaluated at a point (α, β) whose distance from the point $(\alpha_0, 2/\alpha_0)$ in the direction of the normal to the curve $\alpha\beta = 2$ is $\sim e^2$ and equals

$$\tilde{\omega}_i = 1 + \frac{\alpha_0^4 + 4}{4\alpha_0}(\alpha - \alpha_0) + e^2 \frac{3(5\alpha_0^2 - 10\alpha_0 + 4)}{2(3\alpha_0 - 2)(2 - \alpha_0)} + O(e^3) \tag{4.8}$$

The quantity $\tilde{\omega}_i$ for a non-resonance frequency is equal to $\sqrt{3\alpha_0 - 2} + O(e^2)$.

We introduce the resonance detuning $\tilde{\omega}_i = 1 - e^2\chi\sigma_i$. We have the following 2π-periodic motions of the satellite

$$\theta = \frac{\pi}{2} + O(e^2), \quad \psi = \pi + 4e\sqrt{\sigma_i \rho_*} \cos(\nu + \psi_*) + O(e^2) \tag{4.9}$$

for the case when $^2/_3 < \alpha_0 < 1$; in the case when $1 < \alpha_0 < 2$, we simply replace "sin" in the formula for ψ by "cos".

Relations (4.9) define a motion of the satellite in which its axis oscillates in the *OXY* plane of the orbital system of coordinates (Fig. 5a) about its position in the unperturbed motion, with an angular amplitude of the order of e.

Regions 0, 1 and 2 in the plane of the parameters e, α in the neighbourhood of the points $e = 0$, $\alpha = \alpha_0$ ($^2/_3 < \alpha_0 < 1$ or $1 < \alpha_0 < 2$), as shown in Fig. 5(b), have the same meaning as above. If $^2/_3 < \alpha_0 < 1$, the equation of the boundary $\chi = 1$ is

$$\alpha = \alpha_0 + \frac{6\alpha_0(3 - 2\alpha_0)}{(\alpha_0^4 + 4)(2 - \alpha_0)} e^2 + O(e^3)$$

and that of the boundary $\chi = -1$ is

$$\alpha = \alpha_0 - \frac{6\alpha_0(4\alpha_0^2 - 7\alpha_0 + 2)}{(\alpha_0^4 + 4)(3\alpha_0 - 2)(2 - \alpha_0)} e^2 + O(e^3)$$

If $1 < \alpha_0 < 2$, the equations of the boundaries $\chi = 1$ and $\chi = -1$ are interchanged.

Suppose we are given a point (α, β) in a small ($\sim e^2$) neighbourhood of the hyperbola $\alpha\beta = 2$. To determine the number of periodic solutions of the form (4.9) corresponding to this point, we first use the equation of the normal straight line $\beta - 2/\alpha_0 = \alpha_2^0(\alpha - \alpha_0)/2$ to determine the nearest point $(\alpha_0, 2/\alpha_0)$ on the hyperbola. Then, using the equality $\tilde{\omega}_2 = 1 - e^2\chi\sigma_i$ and Eq. (4.8), we find the value of the

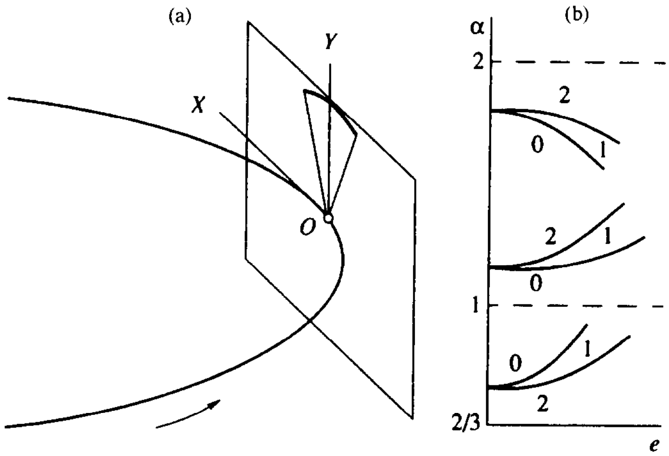

Fig. 5

parameter χ of the model system and draw conclusions about the number and form of the periodic solutions.

In all the resonant cases listed above, the single periodic motion in regions 1 is stable for the majority of initial data; of the two periodic motions in regions 2, one (corresponding to the lower amplitude) is unstable and one (corresponding to the higher amplitude) is stable for the majority of initial data. Condition (2.9) for stable motions is violated only for the resonant case $\omega_i = 1$ and $\alpha\beta = 2$, if for $^2/_3 < \alpha_0 < {}^5/_6$ the parameter χ of the model system takes the value $\chi = \chi_* = 3(2\alpha_0 - 1)/(5 - 6\alpha_0)$.

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